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STRONGLY STABLE STATIONARY SOLUTIONS IN NONLINEAR PROGRAMS.(U)

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Masakazu Kojima

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ABSTRACT

For each continuously twice differentiable map $f = (f_0, f_1, \dots, f_m) : R^n \to R^{1+m}$, we define a nonlinear program:

$$\begin{array}{llll} \text{P1(f)} & \text{minimize} & f_0(x) & \\ & \text{subject to} & f_i(x) = 0 & (1 \leqq i \leqq \ell), \\ & & f_j(x) \leqq 0 & (\ell + 1 \leqq j \leqq m). \end{array}$$

A (Kuhn-Tucker's) stationary solution x to P1(f) is said to be strongly stable if there exists an open neighborhood U of x such that each open neighborhood $V \subset U$ of x contains a stationary solution to a perturbed problem P1(f + g) which is unique in U whenever $g_i(x)$, $\partial g_i(x)/\partial x_j$ and $\partial^2 g_i(x)/\partial x_j \partial x_k$ $(0 \leqq i \leqq m,\ 1 \leqq j \leqq n,\ 1 \leqq k \leqq n)$ are sufficiently small for all x in U. We will give conditions on the gradient vectors and the Hessian matrices of $f_i (0 \leqq i \leqq m)$ which characterize the strong stability. These conditions are then applied to a parametric nonlinear program:

$$\begin{array}{llll} \text{P2(t)} & \text{minimize} & h_0(x,t) & \\ & \text{subject to} & h_i(x,t) = 0 & (1 \leqq i \leqq \ell), \\ & & h_j(x,t) \leqq 0 & (\ell + 1 \leqq j \leqq m), \end{array}$$

where t is a parameter vector varying in a closed subset T of R^q. Let Σ^s be the set of points (x,t) in $R^n \times T$ such that x is a strongly stable stationary solution to P2(t). Under a certain constraint qualification and the continuity and the differentiability of the map $h : R^n \times T \to R^{1+m}$, we will establish that if S is a connected subset of Σ^s and if x^* is a local minimum solution to P2(t^*) for some $(x^*, t^*) \in S$ then x is a local minimum solution to P2(t) for all $(x,t) \in S$. Finally this result is applied to showing some interesting properties of a class of methods developed in the fixed point and complementarity theory.

AMS(MOS) Subject Classification - 90C30, 49B50

Key Words - Nonlinear Program, Stability, Parametric Program, Fixed Point and Complementarity Theory

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SIGNIFICANCE AND EXPLANATION

When we construct mathematical models from practical problems in the field of operations research, economics, engineering, etc., the data which we can utilize usually have uncertainty. We may not get exact data or the data may be changing as time goes. In such a model it is important to take account of the stability of the solution. Here we say that a solution to a model is stable if any slight perturbation to the data yields a small change of the solution.

In this paper we study stability of a mathematical programming model, which involves an objective function to be minimized (or maximized) under certain constraints. We give conditions on the data of the model which characterize the stability. Applications to a mathematical programming model having parameters and a class of computational methods are also discussed.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

STRONGLY STABLE STATIONARY SOLUTIONS IN NONLINEAR PROGRAMS

Masakazu Kojima

1. INTRODUCTION.

Let R^n be the n-dimensional Euclidean space. Throughout the paper we use the ℓ_∞ norm $\|x\| = \max_i |x_i|$ for each $x = (x_1,\ldots,x_n) \in R^n$. Let F be the class of maps $f = (f_0,f_1,\ldots,f_m)$ from R^n into R^{1+m} such that each f_i is continuously twice differentiable, where m is a nonnegative integer. For each $f \in F$ we define the nonlinear program:

Pl(f)

$$\text{minimize } f_0(x)$$
$$\text{subject to } x \in X(f) ,$$

where

$$X(f) = \{x \in R^n : f_i(x) = 0 \quad (1 \leq i \leq \ell) ,$$
$$f_j(x) \leq 0 \quad (\ell + 1 \leq j \leq m)\} .$$

We call $X(f)$ the constraint set to the nonlinear program Pl(f). For each positive number δ and $x \in R^n$, let

$$B_\delta(x) = \{x' \in R^n : \|x' - x\| \leq \delta\} .$$

An $x \in X(f)$ is said to be a local minimum solution to $Pl(f)$ if there is a positive number δ such that

$$f_0(x) \leq f_0(x') \quad \text{for all } x' \in X(f) \cap B_\delta(x) .$$

It is well-known that under an assumption which is called a constraint qualification every local minimum solution to Pl(f) satisfies the Kuhn-Tucker stationary condition (Kuhn and Tucker [20], see also Fiacco and McCormick [15], Mangasarian [23]). We will write the Kuhn-Tucker stationary condition in the form of a system of equations. For each $\alpha \in R$, let

$$\alpha^+ = \max\{0,\alpha\} \text{ and } \alpha^- = \min\{0,\alpha\} .$$

For each continuously twice differentiable map $\psi : R^n \to R$ we use the notations $\nabla\psi(x)$ and $\nabla^2\psi(x)$ for the gradient vector and the $n \times n$ Hessian matrix of ψ at $x \in R^n$. Then the Kuhn-Tucker stationary condition to Pl(f) can be written as the system of equations

(1-1) $$F(x,y) = 0,$$

where

(1-2) $$F(x,y) = \begin{bmatrix} \nabla f_0(x) + \sum_{i=1}^{\ell} y_i \nabla f_i(x) + \sum_{j=\ell+1}^{m} y_j^+ \nabla f_j(x) \\ -f_1(x) \\ \vdots \\ -f_\ell(x) \\ y_{\ell+1}^- - f_{\ell+1}(x) \\ \vdots \\ y_m^- - f_m(x) \end{bmatrix}$$

for all $(x,y) \in R^{n+m}$. If $x \in R^n$ satisfies (1-1) for some $y \in R^m$ then we call x a stationary solution to Pl(f), y a Lagrange multiplier vector associated with x and (x,y) a stationary point to Pl(f). The constraint qualification which we will impose on a stationary (or local minimum) solution x^* to Pl(f) is:

<u>Condition 1.1.</u> The set $\{\nabla f_i(x^*) : f_i(x^*) = 0, 1 \leq i \leq m\}$ is linearly independent.

If a local minimum solution x^* to Pl(f) satisfies Condition 1.1 then it is a stationary solution, i.e., there exist a Lagrange multiplier vector $y^* \in R^m$ such that (x^*,y^*) satisfies (1,1). See Fiacco and McCormick [15]. A stationary (or local minimum) solution x^* to Pl(f) is said to be isolated if there is a positive number such that $B_\delta(x^*)$ contains no stationary (or local minimum) solution to Pl(f) except x^*.

Let x^* be an isolated stationary solution to Pl(f). We will study the existence and uniqueness of a stationary solution to a perturbed problem $Pl(f + g)$ in a small open neighborhood of x^*, where $g \in F$. For each $g \in F$ and a subset U of R^n, let

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$$\mathrm{norm}(g,U) = \sup_{0 \le i \le m} \sup_{x \in U} \max\{|g_i(x)|, \|\nabla g_i(x)\|, \|\nabla^2 g_i(x)\|\} ,$$

where for each $n \times n$ matrix

$$\|A\| = \max\{\|Ax\| : x \in R^n, \|x\| = 1\} .$$

Definition. Let x^* be a stationary solution to P1(f). x^* is strongly stable with respect to a subclass F' of F (abbreviated by s-stable (w.r.t. F')) if for some $\delta^* > 0$ and each $\delta \in (0,\delta^*]$ there exists an $\alpha > 0$ such that whenever $g \in F'$ and $\mathrm{norm}(g, B_{\delta^*}(x^*)) \le \alpha$, $B_\delta(x^*)$ contains a stationary solution to P1(f + g) which is unique in $B_{\delta^*}(x^*)$.

Obviously, if $F_2 \subset F_1 \subset F$ and if a stationary solution x^* to P1(f) is s-stable (w.r.t. F_1) then it is s-stable (w.r.t. F_2). By the definition, we also see that if a stationary solution x^* to P1(f) is not s-stable (w.r.t. a subclass F' of F) then for any positive numbers δ^*, δ, α with $\delta < \delta^*$ there exists a map $g \in F'$ such that $\mathrm{norm}(g, B_{\delta^*}(x^*)) \le \alpha$ and that $B_\delta(x^*)$ contains either no stationary solution to P1(f + g) or more than one stationary solutions to P1(f + g). In this paper, we derive necessary and sufficient conditions for the s-stability (w.r.t. F) and then apply them to showing some topological properties of the stationary solution set to a parametric nonlinear program.

In our discussions, the map $F : R^{n+m} \to R^{n+m}$ defined by (1-2) plays an essential role. Generally, the map F is not continuously differentiable on R^{n+m} but is piecewise continuously differentiable (abbreviated by PC^1) on R^{n+m}. Some definitions and properties on PC^1 maps will be given in Section 2.

In Sections 3 and 4, we will give conditions on the gradients vectors and the Hessian matrices of f_i $(0 \le i \le m)$ which characterize the s-stability (w.r.t. F). We will also show that the s-stability w.r.t. the class F is equivalent to the s-stability with respect to its proper subclass

(1-3) $$F^* = \{g \in F : g_0(x) = \tfrac{1}{2} x^T D x + c^T x, g_i(x) = d_i (1 \le i)$$
$$\text{for some } n \times n \text{ symmetric matrix } D ,$$
$$c \in R^n \text{ and } d = (d_1, \ldots, d_m) \in R^m\} .$$

In Section 5, we will show that a s-stable (w.r.t. F) stationary solution of P1(f) is a local minimum if and only if it satisfies the strong second-order sufficient condition (Robinson [27]) for an isolated local minimum solution.

In Section 6, we will be concerned with the parametric nonlinear program:

P2(t) $(t \in T)$ $$\text{minimize } h_0(x,t)$$
$$\text{subject to } x \in Y(t) ,$$

where

$$Y(t) = \{x \in R^n : h_i(x,t) = 0 \quad (1 \le i \le \ell)$$
$$h_j(x,t) \le 0 \quad (\ell + 1 \le j \le m)\} ,$$

t is a parameter vector varying in a closed subset T of R^q and $h = (h_0, h_1, \ldots, h_m) : R^n \times T \to R^{1+m}$. We assume that $h(\cdot,t) \in F$ for each $t \in T$ and that $h_i(x,t)$, $\partial h_i(x,t)/\partial x_j$ and $\partial^2 h_i(x,t)/\partial x_j \partial x_k$ $(0 \le i \le m, 1 \le j \le n, 1 \le k \le n)$ are continuous with respect to $(x,t) \in R^n \times T$. Let Σ be the set $\{(x,t) \in R^n \times T : x$ is a stationary solution of P2(t)$\}$, and Σ^s the set $\{(x,t) \in \Sigma : x$ is a s-stable (w.r.t. F) stationary solution of P2(t)$\}$. Under the constraint qualification that the set $\{\nabla h_i(x,t) : h_i(x,t) = 0, 1 \le i \le m\}$ is lineary independent for all $(x,t) \in \Sigma^s$, where $\nabla h_i(x,t) = (\partial h_i(x,t)/\partial x_1, \ldots, \partial h_i(x,t)/\partial x_n)^T$, we will prove that if S is a connected set in Σ^s and if x^* is a local minimum solution to P2(t*) for some $(x^*,t^*) \in S$ then x is a local minimum solution to P2(t) for every $(x,t) \in S$.

In Section 7, we will focus our attention to the case where $T = [0,1] \subset R$. In a class of computational methods (Eaves [9, 11], Kojima [17, 19], Lemke [21], Saigal [30], Todd [33]) developed in the fixed point and complementarity theory (see, for example, Allgower and Georg [4], Cottle and Dantzig [7], Eaves [10], Lemke [22], Saigal [30], Todd [32]), the structure of the one dimensional parameter family P2(t) $(t \in T)$ of nonlinear programs is utilized. To solve P1(f), each method of the class artificially

construct a map $h : R^n \times T \to R^{1+m}$ such that $h(x,1) = f(x)$ for all $x \in R^n$, i.e., $P2(1) = P1(f)$ and that the program $P2(0)$ has a trivial stationary solution x^0. Then starting from $(x^0,0) \in \Sigma$, the method traces the connected component $S^0 \subset \Sigma$ containing $(x^0,0)$. Under certain conditions, the method attains an approximation $(\hat{x},1)$ of $(x^*,1) \in S^0$. By the definition, x^* is a stationary solution to $P1(f)$. However, it is not always true that x^* is a local minimum solution to $P1(f)$. Furthermore, it can happen that the value of the objective function f_0 at x^* is greater than the one at the initial solution x^0 even if $x \in X(f)$ for all $(x,t) \in S^0$. As a corollary to the results given in Section 6, we will see that if x^0 is a local minimum solution of $P2(0)$ and if x is a s-stable (w.r.t. Γ) stationary solution of $P2(t)$ for every $(x,t) \in S^0$ then S^0 is a one dimensional curve with the form $\{(x(t),t) : t \in [0,1]\}$ and $x(t)$ is a local minimum solution to $P2(t)$ for every $t \in [0,1]$. It should be noted that if x was not s-stable (w.r.t. Γ) for some $(x,t) \in S^0$, any open neighborhood of x might contain no stationary solution or more than one stationary solution to $P2(t + \varepsilon)$ for every sufficiently small $\varepsilon > 0$. If, in addition, we employ a map $h : R^n \times T \to R^{1+m}$ such that

$$h_0(x,t) = (1 - t)f_0(x) + tg_0(x)$$
$$h_i(x,t) = f_i(x) \quad (1 \leq i \leq m)$$

for all $(x,t) \in R^n \times T$, then we can prove that $f_0(x(t))$ is monotone nonincreasing with respect to $t \in [0,1]$.

Many studies have been made on the stability or the sensitivity of (local) minimum solutions to parametric programs (Berge [5], Dantzig, Folkman and Shapiro [8], Evans and Gould [13], Fiacco [14], Fiacco and McCormick [15], Robinson [25, 26, 27], etc.). They mainly discussed the continuity of the minimum value of the objective function, the continuity of the set of minimum solutions and/or the continuity of an isolated local minimum (or stationary) solution with respect to a small change of the parameter vector. In Robinson [25], a quantitative bound on a variation of an isolated local minimum solution caused by a small change of the parameter vector was given. Under an

additional assumption on the differentiability with respect to the parameter vector, Fiacco [14] provided a method for estimating the variation.

Let $(x,y) \in R^{n+m}$ be a stationary point to $P2(t)$. If $\{i : y_i > 0, \ell+1 \leq i \leq m\} = \{i : h_i(x,t) = 0, \ell + 1 \leq i \leq m\}$, we say that (x,y) satisfies strict complementarity. In the papers [14,25] above, the strict complementarity was assumed at a stationary point whose variation would be investigated so that they could apply the standard implicit function theorem to a system of equations associated with the Kuhn-Tucker stationary condition. If we replace $\nabla f_i(x)$ by $\nabla h_i(x,t) = (\partial h_i(x,t)/\partial x_1, \ldots, \partial h_i(x,t)/\partial x_n)^T$ $(0 \leq i \leq m)$ in (1-2), we will have the map $H(x,y,t) : R^{n+m+1} \to R^{n+m}$ and the system of equations $H(x,y,t) = 0$ which is equivalent to the Kuhn-Tucker stationary condition to $P2(t)$ (see (6-2)). Note that if the strict complementarity holds at a solution (x,y) to $H(x,y,t) = 0$ then the map $H(\cdot,\cdot,t) : R^{n+m} \to R^{n+m}$ is continuously differentiable in some open neighborhood of (x,y). In the case where the strict complementarity does not hold, however, the approach based on the standard implicit function theorem for C^1 maps can not be used. Throughout this paper we do not assume the strict complementarity.

Without the strict complementarity assumption, Robinson [27] recently gave a sufficient condition for the continuity of the variation of an isolated stationary solution to $P2(t)$ with respect to the parameter vector t. His approach is based on the study of generalized equations which include the Kuhn-Tucker stationary condition as a special case (Robinson 26]). On the other hand, our main tools are the degree theory of the continuous maps (see, for example, Ortega and Rheinboldt [24]) and some fundamental results on PC^1 maps (Kojima [18], Whitehead [34]).

2. PRELIMINARIES.

Let P be a subset of R^p and K a collection of a finite number of closed convex polyhedral sets with nonempty interior. K is said to be a subdivision of R^p if

(2-1) the union of all σ in K is P

(2-2) $\sigma_1 \cap \sigma_2$ is a common face of σ_1 and σ_2 for every pair σ_1 and σ_2 in K with $\sigma_1 \cap \sigma_2 \neq \phi$.

We call each $\sigma \in K$ a piece of K. When K is a subdivision of P, we will write $P = |K|$. A PL (piecewise linear) map $\phi : |K| \to R^p$ is a continuous map such that the restriction $\phi|\sigma$ of the map ϕ to each piece $\sigma \in K$ is affine, i.e., there exist a $p \times p$ matrix $A(\sigma)$ and an $a(\sigma) \in R^p$ such that

$$\phi(z) = A(\sigma)z + a(\sigma) \quad \text{for every } z \in \sigma .$$

A PC^1 (piecewise continuously differentiable) map $\phi : |K| \to R^p$ is a continuous map such that for each piece σ of K there exist an open set $U \supset \sigma$ and a continuously differentiable map $\phi' : U \to R^p$ such that $\phi(z) = \phi'(z)$ for all $z \in \sigma$. Obviously, a PL map $\phi : |K| \to R^p$ is PC^1. We will use the symbol $D\phi(z;\sigma)$ for the Jacobian matrix of the restriction $\phi|\sigma$ of a PC^1 map $\phi : |K| \to R^p$ to a piece σ of K at $z \in \sigma$.

Now we shall show that the map $F : R^{n+m} \to R^{n+m}$ defined by (1-2) is PC^1. Let J° denote the index set $\{\ell+1, \ldots, m\}$. For each $J \subset J^\circ$, define

(2-3) $$\tau(J) = \{(x,y) \in R^{n+m} : y_i \geqq 0 (i \in J), y_j \leqq 0 (j \in J^\circ \backslash J)\}$$

where $J^\circ \backslash J = \{j \in J^\circ : j \notin J\}$. We include the two cases $J = \phi$ and $J = J^\circ$;

$$\tau(\phi) = \{(x,y) \in R^{n+m} : y_j \leqq 0 (j \in J^\circ)\}$$

$$\tau(J^\circ) = \{(x,y) \in R^{n+m} : y_j \geqq 0 (j \in J^\circ)\} .$$

Let

(2-4) $$K^\circ = \{\tau(J) : J \subset J^\circ\} .$$

Then K° is a subdivision of R^{n+m}, i.e., $R^{n+m} = |K^\circ|$. It is easily verified that F is PC^1 on the subdivision K°.

Let $\phi^k : |K| \to R^p$ be PC^1 $(k = 1,2)$, where $P = |K| \subset R^p$, and ε, ρ be positive numbers. If

$$\|\phi^2(z) - \phi^1(z)\| \leqq \varepsilon \quad \text{for all } z \in |K|$$

and

$$\|(D\phi^2(z;\sigma) - D\phi^1(z;\sigma))u\| \leqq \rho \|D\phi^1(z;\sigma)u\|$$

$$\text{for all } z \in \sigma, \ \sigma \in K \text{ and } u \in R^p \text{ with } \|u\| = 1 ,$$

we say that ϕ^2 is an (ε,ρ)-approximation of ϕ^1. See Kojima [18] or Whitehead [34] for more general definition.

Let $\phi : |K| \to R^p$ be PC^1, where $P = |K| \subset R^p$. For each $z^\circ \in |K|$, let

(2-5) $$M(z^\circ,K) = \{\sigma \in K : z^\circ \in \sigma\} .$$

It is easily verified that

$$D\phi(z^\circ;\tau)(z - z^\circ) = D\phi(z^\circ;\sigma)(z - z^\circ)$$

$$\text{if } \tau,\sigma \in M(z^\circ,K) \text{ and } z \in \tau \cap \sigma .$$

Hence we can consistently define a PL map $\phi^{z^\circ} : |M(z^\circ,K)| \to R^p$ by

$$\phi^{z^\circ}(z) = \phi(z^\circ) + D\phi(z^\circ;\sigma)(z - z^\circ) \quad \text{for every } z \in \sigma \text{ and } \sigma \in M(z^\circ,K) .$$

The map $\phi^{z^\circ} : |M(z^\circ,K)| \to R^p$ can be regarded as a PL approximation of $\phi : |K| \to R^p$ at the point $z^\circ \in |K|$ with the use of the Jacobian matirx. A PC^1 map $\phi : |K| \to R^p$ is said to be locally nonsingular at $z^\circ \in |K|$ if the PL map $\phi^{z^\circ} : |M(z^\circ,K)| \to R^p$ is one-to-one. A nonsingular PC^1 map $\phi : |K| \to R^p$ is a PC^1 map which is locally nonsingular at every $z^\circ \in |K|$.

Suppose (x°,y°) is a stationary point to P1(f) (i.e., a solution to (1-1)) and that Condition 1.1 holds. In Section 3, we will show that $F : |K^\circ| \to R^{n+m}$ is locally nonsingular at $z^\circ = (x^\circ,y^\circ)$ if and only if

$$\det DF(z^\circ;\sigma) > 0 \quad \text{for all } \sigma \in M(z^\circ,K^\circ)$$

or

$$\det DF(z^\circ;\sigma) < 0 \quad \text{for all } \sigma \in M(z^\circ,K^\circ) .$$

In Section 4, we will establish that x° is s-stable (w.r.t. f) if and only if $z^\circ = (x^\circ,y^\circ)$, is a locally nonsingular point of the map $F : |K^\circ| \to R^{n+m}$.

We now state some fundamental lemmas on PC^1 maps which we will use later. Throughout the lemmas below, K denotes a subdivision of a subset P of R^p.

Lemma 2.1. Assume that $P = |K|$ is compact and that $\psi : |K| \to R^p$ is a one-to-one and nonsingular PC^1 map. Then there exist positive numbers ε, ρ such that every (ε,ρ)-approximation to $\psi : |K| \to R^p$ is one-to-one and nonsingular.

Proof. See Theorem 3 of Whitehead [34].

Lemma 2.2. Let $\psi : |K| \to R^p$ be PC^1. Assume that

(2-6) a point z^* lies in the interior of P,

(2-7) $\psi : |K| \to R^p$ is locally nonsingular at z^*.

Then there is a positive number δ such that

(2-8) $\psi : |K| \to R^p$ is locally nonsingular at every $z \in B_\delta(z^*) \cap P$,

(2-9) ψ is one-to-one in $B_\delta(z^*) \cap P$.

Proof. See Lemmas 2-10 and 2-11 of Kojima [18].

For each subset U of R^p, we use the notations $\mathrm{int}\, U$, $\mathrm{c\ell}\, U$ and $\mathrm{bd}\, U$ for the interior of U, the closure of U and the boundary of U, respectively. Let U be an open subset of R^p and $\psi : \mathrm{c\ell}\, U \to R^p$ be continuous. Suppose that $\psi(z) \neq c$ for all $z \in \mathrm{bd}\, U$. We define $\deg(\psi,U,c)$ to be the degree of the map ψ at c with respect to U (see, for example, Ortega and Rheinboldt [24]).

Lemma 2.3. Let $\psi : |K| \to R^p$ be PC^1 and z^* an interior point of $P = |K|$. Assume that $B_{\delta^*}(z^*) \subset |K|$ for some positive δ^* and that ψ is one-to-one in $B_{\delta^*}(z^*)$. Then either

(2-10) $\deg(\psi, \mathrm{int}\, B_\delta(z^*), \psi(z^*)) = +1$ for every $\delta \in (0,\delta^*]$

or

(2-11) $\deg(\psi, \mathrm{int}\, B_\delta(z^*), \psi(z^*)) = -1$ for every $\delta \in (0,\delta^*]$.

If (2-10) (or (2-11)) holds then

$$\det D\psi(z;\sigma) \geq 0 \quad (\text{or} \leq 0)$$

$$\text{for all } z \in \sigma \cap \mathrm{int}\, B_{\delta^*}(z^*) \text{ and } \sigma \in K .$$

Proof. Since $B_{\delta^*}(z^*)$ is a compact subset of R^p, ψ maps $B_{\delta^*}(z^*)$ onto $\psi(B_{\delta^*}(z^*))$ homeomorphically. Hence either (2-10) or (2-11) holds (see, for example, Chapter XVI,

§4 of Alexandroff [3]). On the other hand, by the invariance theorem of domain (Lemma 3.9 of Eilenberg and Steenrod [11]), we see that $\psi(\mathrm{int}\, B_\delta(z^*))$ is an open subset of R^p for each $\delta \in (0,\delta^*]$. We assume that (2-10) holds, and show that

$$\det D\psi(z;\sigma) \geq 0 \text{ for all } z \in \sigma \cap \mathrm{int}\, B_{\delta^*}(z^*) \text{ and } \sigma \in K .$$

Let $z \in \sigma \cap \mathrm{int}\, B_{\delta^*}(z^*)$ and $\det D\psi(z;\sigma) \neq 0$. Then there is a $z' \in \mathrm{int}\, \sigma \cap \mathrm{int}\, B_{\delta^*}(z^*)$ such that $\mathrm{sign} \det D\psi(z;\sigma) = \mathrm{sign} \det D\psi(z';\sigma) \neq 0$. Thus it suffices to show that $\det D\psi(z';\sigma) > 0$. Note that ψ is continuously differentiable in some open neighborhood of z'. Since $\mathrm{int}\, B_{\delta^*}(z^*)$ is a convex subset of R^p and ψ is one-to-one in $B_{\delta^*}(z^*)$, the image of the line segment $\{(1-t)z^* + tz' : 0 \leq t \leq 1\} \subset \mathrm{int}\, B_{\delta^*}(z^*)$ under the map ψ does not intersect with $\psi(\mathrm{bd}\, B_{\delta^*}(z^*))$. By the homotopy invariance theorem (6.2.2 of Ortega and Rheinboldt [24]), we obtain

$$\begin{aligned} 1 &= \deg(\psi, \mathrm{int}\, B_{\delta^*}(z^*), \psi(z^*)) \\ &= \deg(\psi, \mathrm{int}\, B_{\delta^*}(z^*), \psi(z')) \\ &= \mathrm{sign} \det D\psi(z';\sigma) . \end{aligned}$$

Q.E.D.

3. LOCAL NONSINGULARITY OF THE MAP $F : |K^*| \to R^{n+m}$.

In this section we give a necessary and sufficient condition for the PC^1 map $F : |K^*| \to R^{n+m}$ defined by (1-2) to be locally nonsingular at a $z^* \in |K^*|$.

If $w^1,\ldots,w^k$ are p-dimensional vectors, we denote by $\langle w^1,\ldots,w^k\rangle$ the closed convex polyhedral cone spanned by them, i.e.,

$$\langle w^1,\ldots,w^k\rangle = \{\sum_{i=1}^{k} \lambda_i w^i : \lambda_i \geq 0\ (1 \leq i \leq k)\} .$$

Lemma 3.1 (Samelson, Thrall and Wesler [31]).
Let $u^1,\ldots,u^p$, $v^1,\ldots,v^p$ be p-dimensional vectors. Define

$$\Pi = \{\langle w^1,\ldots,w^p\rangle : w^i = u^i \text{ or } v^i\ (i = 1,\ldots,p)\} .$$

Suppose that $\det[u^1,\ldots,u^p] > 0$. Then Π is a subdivision of R^p if and only if for every $\langle w^1,\ldots,w^p\rangle \in \Pi$

$$\operatorname{sign} \det[w^1,\ldots,w^p] = (-1)^s ,$$

where s is the number of v^i's among $w^1,\ldots,w^p$.

Corollary 3.2. Let Π be the collection of all the orthants of R^p, i.e.,

$$\Pi = \{\langle w^1,\ldots,w^p\rangle : w^i = e^i \text{ or } -e^i\ (1 \leq i \leq p)\} ,$$

where e^i denotes the i-th unit vector in R^p. Let $\phi : |\Pi| \to R^p$ be PL. Then ϕ is one-to-one if and only if either

$$\det D\phi(0;\sigma) > 0 \quad \text{for all } \sigma \in \Pi$$

or

$$\det D\phi(0;\sigma) < 0 \quad \text{for all } \sigma \in \Pi .$$

Proof. We first show the "only if" part. Suppose that ϕ is one-to-one on $|\Pi| = R^p$. Since $\phi : |\Pi| \to R^p$ is piecewise linear, we have

$$\phi(x) = D\phi(0;\sigma)x + \phi(0) \quad \text{for every } x \in \sigma \in \Pi .$$

Hence

$$\det D\phi(0;\sigma) \neq 0 \quad \text{for all } \sigma \in \Pi ,$$

and the desired result follows from Lemma 2.3.

We now show the "if" part. Performing an appropriate linear transformation to the map ϕ if necessary, we may assume that $\phi(0) = 0$,

$$\det D\phi(0;\sigma) > 0 \quad \text{for all } \sigma \in \Pi$$

and that

$$D\phi(0;\sigma^-) = [e^1,\ldots,e^p] ,$$

where $\sigma^- = \langle -e^1,\ldots,-e^p\rangle$. Then we have

$$\phi(z) = D\phi(0;\sigma)z \quad \text{for every } z \in \sigma \in \Pi .$$

Let $\sigma^+ = \langle e^1,\ldots,e^p\rangle$ and $D\phi(0,\sigma^+) = [u^1,\ldots,u^p]$. Then, for every $\sigma \in \Pi$, we have

$$\phi(e^i) = D\phi(0;\sigma)e^i = D\phi(0;\sigma^+)e^i = u^i \quad \text{if } e^i \in \sigma$$

and

$$\phi(-e^i) = D\phi(0;\sigma)(-e^i) = D\phi(0;\sigma^-)(-e^i) = -e^i \quad \text{if } -e^i \in \sigma .$$

Hence, for every $\sigma \in \Pi$,

$$D\phi(0;\sigma)e^i = \begin{cases} u^i & \text{if } e^i \in \sigma \\ e^i & \text{if } -e^i \in \sigma . \end{cases}$$

Recall that $\det D\phi(0;\sigma) > 0$ for all $\sigma \in \Pi$. By Lemma 3.1, the collection

$$\{\langle w^1,\ldots,w^p\rangle : w^i = u^i \text{ or } -e^i\ (1 \leq i \leq n)\}$$

subdivides R^p. This is equivalent to that $\phi : |\Pi| \to R^p$ is one-to-one. Q.E.D.

Now we are ready to derive a necessary and sufficient condition for the local nonsingularity of the map $F : |K^*| \to R^{n+m}$ at a $z^* \in |K^*|$. Let $z^* = (x^*,y^*) \in |K^*| = R^{n+m}$. Let

$$J_0 = \{j : y_j^* > 0, \ell + 1 \leq j \leq m\} ,$$

(3-1) and

$$J_1 = \{j : y_j^* \geq 0, \ell + 1 \leq j \leq m\} .$$

Obviously, we see $J_0 \subset J_1$ and

$$M(z^*,K^*) = \{\tau(J) : J_0 \subset J \subset J_1\} ,$$

where $M(z^*,K^*)$ denotes the collection of pieces $\tau(J) \in K^*$ which contain z^* and $\tau(J)$ is defined by (2-3). By the definition of the local nonsingularity,

$F : |K^*| \to R^{n+m}$ is locally nonsingular at z^* if and only if the map $\hat{F} : |M(z^*;K^*)| \to R^{n+m}$ defined by

$$\hat{F}(z) = F(z^*) + DF(z^*;\tau(J))(z - z^*) \quad \text{for every } z \in \tau(J) \in M(z^*,K^*)$$

is one-to-one. For each J such that $J_0 \subset J \subset J_1$, let

$$\bar{\tau}(J) = \{\lambda(z - z^*) : z \in \tau(J), \lambda \geq 0\}$$

or equivalently

$$\bar{\tau}(J) = \{(x,y) \in R^{n+m} : y_j \geq 0 (j \in J \backslash J_0) \text{ and } y_k \leq 0 (k \in J_1 \backslash J)\} ,$$

and

$$\bar{M} = \{\bar{\tau}(J) : J_0 \subset J \subset J_1\} .$$

Then $\bar{M}$ is a subdivision of R^{n+m}. It can be easily shown that $\hat{F} : |M(z^*,K^*)| \to R^{n+m}$ is one-to-one if and only if the PL map $\bar{F} : |\bar{M}| \to R^{n+m}$ defined by

$$\bar{F}(z) = DF(z^*;\tau(J))z \quad \text{for every } z \in \bar{\tau}(J) \in \bar{M}$$

is one-to-one. Since each $\tau(J) \in \bar{M}$ is the union of some orthants of R^{n+m}, we can regard $\bar{F} : R^{n+m} \to R^{n+m}$ as a PL map on the subdivision N consisting of all the orthants of R^{n+m}. Thus, by Corollary 3.2, we obtain:

<u>Theorem 3.3.</u> Let $z^* = (x^*,y^*) \in |K^*| = R^{n+m}$, and let J_0, J_1 be the index set defined by (3-1). Then $F : |K^*| \to R^{n+m}$ is locally nonsingular at z^* if and only if either

(3-2) $\qquad \det DF(z^*;\tau(J)) > 0$ for all J such that $J_0 \subset J \subset J_1$

or

(3-3) $\qquad \det DF(z^*;\tau(J)) < 0$ for all J such that $J_0 \subset J \subset J_1$.

The remainder of this section is devoted to derive conditions which characterize (3-2) (or (3-3)) in terms of the gradients vectors and the Hessian matrix of the maps $f_i : R^n \to R$ $(0 \leq i \leq m)$ at x^*.

<u>Lemma 3.4.</u> Let D be an $n \times n$ symmetric matrix and B an $n \times p$ matrix with rank $B = p$, where $1 \leq p \leq n$. Assume that $\det B^TDB = 0$. Then there exist $n \times n$ symmetric matrices Q^+ and Q^- such that, for every $\gamma > 0$,

(3-4) $$\det B^T(D + \gamma Q^+)B > 0$$

and

(3-4)' $$\det B^T(D + \gamma Q^-)B < 0 .$$

<u>Proof.</u> Since B^TDB is a $p \times p$ symmetrix matrix with $\det B^TDB = 0$, we can find a $p \times p$ nonsingular matrix U such that

$$B^TDB = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{bmatrix} U^T ,$$

$$\lambda_i = 0 \ (1 \leq i \leq k) \quad \text{and} \quad R \ni \lambda_j \neq 0 \ (k+1 \leq j \leq p) ,$$

where the i-th column of U is an eigenvector of the matrix B^TDB associated with the eigenvalue λ_i and k is a positive integer not greater than p. Choose $\nu_i \in \{-1,0,1\}$ $(1 \leq i \leq p)$ such that

(3-5) $$\nu_1 \cdots \nu_k \lambda_{k+1} \cdots \lambda_p > 0 \quad \text{and} \quad \nu_j = 0 \ (k+1 \leq j \leq p) .$$

Define the $p \times p$ matrix

$$V^+ = U \begin{bmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_p \end{bmatrix} U^T .$$

By the construction, we have, for every $\gamma > 0$

$$\begin{aligned} &\det(B^TDB + \gamma V^+) \\ &= \det\left(U \begin{bmatrix} \lambda_1 + \gamma\nu_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p + \gamma\nu_p \end{bmatrix} U^T\right) \\ &= \gamma^k (\det U)^2 \nu_1 \cdots \nu_k \lambda_{k+1} \cdots \lambda_p > 0 . \end{aligned}$$

On the other hand, it follows from rank $B = p$ that the $n \times p$ matrix B contains a set of p linearly independent rows. For simplicity of notations, we assume that the set of the first p rows is linearly independent, and partition B as follows:

$$B = \begin{array}{c} \\ \end{array} \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} \begin{array}{l} p \\ n-p \end{array}$$

Define the n n matrix Q^+ by

$$Q^+ = \left[\begin{array}{c|c} (B_1^{-1})^T V^+ B_1^{-1} & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} p \\ n-p \end{array} .$$

Obviously Q^+ is symmetric. By a simple calculation we see that Q^+ satisfies (3-4) for every $\gamma > 0$. The existence of an $n \times n$ symmetric matrix Q^- satisfying (3-4)' can be shown similarly if we replace (3-5) by

$$\mu_1 \cdots \mu_k \lambda_{k+1} \cdots \lambda_p < 0 \quad \text{and} \quad \mu_j = 0 \ (k+1 \leq j \leq p) .$$

Q.E.D.

We introduce some notations. For each $z = (x,y) \in R^n$, let

(3-6) $$L(z) = \nabla^2 f_0(x) + \sum_{i=1}^{\ell} y_i \nabla^2 f_i(x) + \sum_{i=\ell+1}^{m} y_i^+ \nabla^2 f_i(x) .$$

We can express the Jacobian matrix $DF(z;\tau(J))$ for each $\tau(J) \in M(z,K^*)$ in terms of $L(z)$ and $\nabla f_i(x)$ $(1 \leq i \leq m)$. For simplicity, we assume $J = \{\ell+1,\dots,k\}$ for some $k \leq m$. Then

(3-7) $$DF(z;\tau(J)) = \begin{array}{c} \begin{array}{ccc} n & k & m-k \end{array} \\ \left[\begin{array}{c|c|c} L(z) & \nabla f_1(x),\dots,\nabla f_k(x) & 0 \\ \hline -\nabla f_1(x)^T & & \\ \vdots & 0 & 0 \\ -\nabla f_k(x)^T & & \\ \hline \vdots & 0 & E \\ -\nabla f_m(x)^T & & \end{array}\right] \end{array} \begin{array}{c} n \\ k \\ m-k \end{array}$$

where E denotes the $(m-k) \times (m-k)$ identity matrix. Hence

(3-8) $$\det DF(z;\tau(J)) = \det \begin{bmatrix} L(z) & A \\ -A^T & 0 \end{bmatrix} ,$$

where

(3-9) $$A = [\nabla f_1(x),\dots,\nabla f_k(x)] .$$

Let D be an $n \times n$ symmetric matrix and W a subspace of R^n with $\dim W = p \geq 1$. We say that the matrix D has a positive (negative or zero) determinant if $\det B^T DB > 0$ (< 0 or $= 0$), where B is an $n \times p$ matrix whose columns form a basis of W. We also say that the matrix D is positive definite (or positive semi-definite) on the subspace W if the $p \times p$ matrix $B^T DB$ is positive definite (or positive semi-definite). For convenience, we assume that each $n \times n$ matrix has a positive determinant and is positive definite on the zero dimensional subspace $W = \{0\}$.

<u>Theorem 3.5.</u> Let $z^* = (x^*,y^*) \in \tau(J)$ for some $J \subset \{\ell+1,\dots,m\}$. If the set

(3-10) $$\{\nabla f_i(x^*) : i \in \{1,\dots,\ell\} \cup J\}$$

is linearly dependent then $\det DF(z^*;\tau(J)) = 0$. In the case where the set (3-10) is linearly independent,

$$\det DF(z^*;\tau(J)) > 0 \ (= 0 \ \text{or} \ < 0)$$

if and only if the $n \times n$ matrix $L(z^*)$ has a positive (zero or negative) determinant on the space $W = \{w \in R^n : \nabla f_i(x^*)^T w = 0 \ (i \in \{1,2,\dots,\ell\} \cup J)\}$.

<u>Proof.</u> For simplicity, we assume that $J = \{\ell+1,\dots,k\}$. Then (3-8) holds where A is the $n \times k$ matrix defined by (3-9). The first assertion of the theorem follows immediately. Suppose that A has rank k. If $k = n$ then the dimension of the subspace W is zero and $\det DF(z^*;\tau(J)) = (\det A)^2 > 0$; hence we obtain the desired result. Assume that $k < n$. Choose an $n \times (n-k)$ matrix B such that $B^T A$ is the $(n-k) \times k$ zero matrix and that $B^T B$ is the $k \times k$ identity matrix. Then the set of columns of B forms a basis of the subspace W. Let $D = L(z^*)$ and

$$C = [A \ \ DB] .$$

We first deal with the case $\det D \neq 0$. It follows from the identity

$$\begin{bmatrix} D^{-1} & 0 \\ A^T D^{-1} & E \end{bmatrix} \begin{bmatrix} D & A \\ -A^T & 0 \end{bmatrix} = \begin{bmatrix} E & D^{-1}A \\ 0 & A^T D^{-1} A \end{bmatrix}$$

that

(3-11) $$\det D^{-1} \det \begin{bmatrix} D & A \\ -A^T & 0 \end{bmatrix} = \det A^T D^{-1} A ,$$

where E denotes the identity matrix with an appropriate size. We also see

$$C^T D^{-1} C = \begin{bmatrix} A^T D^{-1} A & 0 \\ 0 & B^T DB \end{bmatrix} .$$

Hence

$$(\det C)^2 (\det D^{-1}) = (\det A^T D^{-1} A)(\det B^T DB) .$$

Substituting (3-11) into the above equality, we have

(3-12) $$(\det C)^2 = \det \begin{bmatrix} D & A \\ -A^T & 0 \end{bmatrix} (\det B^T DB) .$$

When $\det D = 0$ we have

(3-12)' $$(\det C(\varepsilon))^2 = \det \begin{bmatrix} D(\varepsilon) & A \\ -A^T & 0 \end{bmatrix} (\det B^T D(\varepsilon) B)$$

for every sufficiently small $\varepsilon > 0$, where

$$D(\varepsilon) = D + \varepsilon E,\ C(\varepsilon) = [A\ D(\varepsilon)B]$$

Note that $\det D(\varepsilon) \neq 0$ for every sufficiently small $\varepsilon > 0$. Taking the limit as $\varepsilon \to 0$ in (3-12)', we obtain (3-12). Therefore (3-12) holds even if $\det D = 0$.

On the other hand, we see

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} \begin{bmatrix} A & DB \end{bmatrix} = \begin{bmatrix} A^T A & A^T DB \\ 0 & B^T DB \end{bmatrix},$$

which implies

$$(\det [A\ B])(\det C) = (\det A^T A)(\det B^T DB)\ .$$

Since $\det [A\ B] \neq 0$ and $(\det A^T A) \neq 0$, we have that $\det C = 0$ if and only if $\det B^T DB = 0$. Therefore, by (3-12), we obtain

(3-13) $$\det \begin{bmatrix} D & A \\ -A^T & 0 \end{bmatrix} > 0 \ \ (\text{or} < 0) \ \text{ if } \ \det B^T DB > 0 \ \ (\text{or} < 0).$$

What we have left to show is

$$\det \begin{bmatrix} D & A \\ -A^T & 0 \end{bmatrix} = 0 \ \text{ if } \ \det B^T DB = 0\ .$$

Assume $\det B^T DB = 0$. By Lemma 3.4, there exist $n \times n$ symmetric matrices Q^+ and Q^- satisfying (3-4) and (3-5) for every $\gamma > 0$. Replacing D by $D + \gamma Q^+$ (or $D + \gamma Q^-$) in (3-13) and taking the limit as $\gamma \to 0$, we obtain

$$\det \begin{bmatrix} D & A \\ -A^T & 0 \end{bmatrix} \geqq 0 \ \ (\text{or} \leqq 0)\ .$$

Q.E.D.

As a direct consequence of Theorems 3.3 and 3.5, we obtain the following result:

<u>Corollary 3.6.</u> Let $z^* = (x^*, y^*)$, J_0 and J_1 be as in Theorem 3.3. Then $F : |K^*| \to R^{n+m}$ is locally nonsingular at z^* if and only if

(3-14) the set $\{\nabla f_j(x^*) : j \in \{1,\ldots,\ell\} \cup J_1\}$ is linearly independent

and

(3-15) for all J such that $J_0 \subset J \subset J_1$ the $n \times n$ matrix $L(z^*)$ has a positive (or negative) determinant on the subspace $\{w \in R^n : \nabla f_i(x^*)^T w = 0\ (i \in \{1,2,\ldots,\ell\} \cup J)\}$.

4. MAIN RESULTS.

For each perturbation $g \in \mathcal{F}$ to the problem $\mathrm{Pl}(f)$, we define the PC^1 map $G : |K^*| \to R^{n+m}$ as follows:

$$G(x,y) = \begin{pmatrix} \nabla g_0(x) + \sum_{i=0}^{\ell} y_i \nabla g_i(x) + \sum_{j=\ell+1}^{m} y_i^+ \nabla g_j(x) \\ -g_1(x) \\ \vdots \\ -g_\ell(x) \\ -g_{\ell+1}(x) \\ \vdots \\ -g_m(x) \end{pmatrix} \tag{4-1}$$

for every $(x,y) \in R^{n+m}$. Then the Kuhn-Tucker stationary condition to the perturbed problem $\mathrm{Pl}(f+g)$ can be written as

$$F(x,y) + G(x,y) = 0 \tag{4-2}$$

Theorem 4.1. Let $z^* = (x^*,y^*)$ be a stationary point to $\mathrm{Pl}(f)$. Assume that F is locally nonsingular at z^*. Then x^* is a s-stable (w.r.t. $\mathcal{F}$) stationary solution to $\mathrm{Pl}(f)$.

Proof. Let $I^* = \{i : f_i(x^*) = 0,\ 1 \leq i \leq m\}$. By Corollary 3.6, the set $\{\nabla f_i(x^*) : i \in I^*\}$ is linearly independent, i.e., Condition 1.1 holds. For simplicity, we assume that $I^* = \{1,2,\ldots,k\}$. By the continuity of f_i $(1 \leq i \leq m)$ at x^*, we can take positive numbers δ_0 and α_0 such that if $x \in B_{\delta_0}(x^*)$,

$$|g_i(x)| \leq \alpha_0,\ \|\nabla g_i(x)\| \leq \alpha_0 \ (1 \leq i \leq m)$$

then

$$\{i : f_i(x) + g_i(x) = 0, 1 \leq i \leq m\} \subset \{1,\ldots,k\}. \tag{4-3}$$

On the other hand, by Lemma 2.2, there is a positive number $\delta_1 \leq \delta_0$ such that F is locally nonsingular at every $z \in B_{\delta_1}(z^*)$ and that F is one-to-one in $B_{\delta_1}(z^*)$. If $x \in B_{\delta_1}(x^*)$, $|g_i(x)| \leq \alpha_0, \|\nabla g_i(x)\| \leq \alpha_0$ $(1 \leq i \leq m)$ and (x,y) satisfies (4-2) then (4-3) and

$$-(\nabla f_0(x) + \nabla g_0(x)) = \sum_{i=1}^{\ell} y_i(\nabla f_i(x) + \nabla g_i(x)) + \sum_{i=\ell+1}^{k} y_i^+(\nabla f_i(x) + \nabla g_i(x)),$$

$$y_i^- = (f_i(x) + g_i(x)) < 0 \ (k+1 \leq i \leq m)$$

hold. Since the set of vectors $\nabla f_i(x^*)$ $(1 \leq i \leq k)$ is linearly independent, there is a positive numbers $\delta_2 \leq \delta_1$ and $\alpha_1 \leq \alpha_0$ such that if $x \in B_{\delta_2}(x^*)$, $|g_i(x)| \leq \alpha_1, \|\nabla g_i(x)\| \leq \alpha_1$ $(0 \leq i \leq m)$ and (x,y) satisfies (4-2) then $y \in B_{\delta_1}(y^*)$. Therefore we have shown the existence of positive numbers δ_1, δ_2 and α_1 such that

(4-4) F is one-to-one on $B_{\delta_2}(x^*) \times B_{\delta_1}(y^*)$,

(4-5) F is locally nonsingular at every $z \in B_{\delta_2}(x^*) \times B_{\delta_1}(y^*)$,

(4-6) if $\mathrm{norm}\,(g, B_{\delta_2}(x^*)) \leq \alpha_1, x \in B_{\delta_2}(x^*)$ and (x,y) is a solution to (4-2) then $y \in B_{\delta_1}(y^*)$.

What we have left to show is that for every $\delta \in (0,\delta_2]$ there is an $\alpha \in (0,\alpha_1]$ such that whenever $\mathrm{norm}\,(g, B_{\delta_2}(x^*)) \leq \alpha$, $B_\delta(x^*) \times B_\delta(y^*)$ contains a solution to (4-2) which is unique in $B_{\delta_2}(x^*) \times B_{\delta_1}(y^*)$. Let $P = B_{\delta_2}(x^*) \times B_{\delta_1}(y^*)$. Note that the restriction $F|P$ of the map F to the polyhedral set P is PC^1. By (4-4) and (4-5), and by Lemma 2.1, there are positive numbers ϵ and ρ such that if the map G given by (4-1) satisfies

$$\|G(z)\| \leq \epsilon \text{ for every } z = (x,y) \in P \tag{4-7}$$

and

$$\|DG(z;\sigma)w\| \leq \rho \|DF(z;\sigma)w\| \tag{4-8}$$

$$\text{for every } z \in \sigma \cap P,\ \sigma \in K^* \text{ and } w \in R^{n+m} \text{ with } \|w\| = 1,$$

then $F + G$ is one-to-one in P. Define

$$\gamma = \min\{\|DF(z;\sigma)w\| : z \in \sigma \cap P, \sigma \in K, \|w\| = 1\}.$$

It follows from (4-5) that $\gamma > 0$. Choose a positive number $\alpha_2 \leq \alpha_1$ such that if $\mathrm{norm}\,(g, B_{\delta_2}(x^*)) \leq \alpha_2$ then the corresponding $G : |K^*| \to R^{n+m}$ defined by (4-1) satisfies (4-7) and

$$\|DG(z;\sigma)w\| \leq \rho\gamma \text{ for all } z \in \sigma \cap P,\ \sigma \in K \text{ and } w \in R^{n+m} \text{ with } \|w\| = 1;$$

hence G satisfies (4-8) and $F+G$ is one-to-one in P. Thus $\mathrm{norm}\,(g,B_{\delta_2}(x^*)) \leq \alpha_2$ implies that P contains at most one solution to (4-2).

Finally we prove that for each positive number $\delta \leq \delta_2$ there is a positive number $\alpha \leq \alpha_2$ such that $B_\delta(z^*) = B_\delta(x^*) \times B_\delta(y^*)$ contains a solution to (4-2) whenever $\rho(g,B_{\delta_2}(x^*)) \leq \alpha$. By Lemma 2.3 and the homotopy invariance theorem (6.2.2 of Ortega and Rheinboldt [24]), we can find a positive number $\alpha < \alpha_2$ such if $\mathrm{norm}\,(g,B_{\delta_2}(x^*)) \leq \alpha$ then

$$\deg(F+G,B_\delta(z^*),0) = \deg(F,B_\delta(z^*),0) = +1 \text{ or } -1 ,$$

which implies that $B_\delta(z^*)$ contains a solution to (4-2) (see Kronecker Theorem, 6.3.1 of Ortega and Rheinboldt [24]). Q.E.D.

<u>Theorem 4.2.</u> Let $z^* = (x^*,y^*)$ be a stationary point to Pl(f). Assume that the Condition 1.1 holds and that x^* is a s-stable (w.r.t. F^*) stationary solution to Pl(f). where F^* is given by (1-3). Then F is locally nonsingular at $z^* = (x^*,y^*)$.

<u>Proof.</u> For simplicity of notations, we assume that $\{i : f_i(x^*) = 0, 1 \leq i \leq m\} = \{1,2,\ldots,k_1\}$ for some k_1. It follows from the first assumption that the set of vectors $\nabla f_i(x^*)$ $(1 \leq i \leq k_1)$ is linearly independent. By the continuity of f_i and ∇f_i at $x^*(1 \leq i \leq m)$, we can take positive numbers δ_0 and α_0 such that if $x \in B_{\delta_0}(x^*)$, $|g_i(x)| \leq \alpha_0$ and $\|\nabla g_i(x)\| \leq \alpha_0$ $(1 \leq i \leq m)$ then

$$\{i : f_i(x) + g_i(x) = 0, 1 \leq i \leq m\} \subset \{1,\ldots,k_1\}$$

and the set of the vectors $\nabla f_i(x) + \nabla g_i(x)$ $(1 \leq i \leq k_1)$ is linearly independent. By the second assumption, for some $\delta^* > 0$ and each positive $\delta \leq \min\{\delta_0,\delta^*\}$ there is a positive number $\alpha \leq \alpha_0$ such that if $\mathrm{norm}\,(g,B_{\delta^*}(x^*)) \leq \alpha$ then $B_\delta(x^*) \times R^m$ contains a solution to (4-2) which is unique in $B_{\delta^*}(x^*) \times R^m$. Taking

$$g_0(x) = -c^T x ,$$

$$g_i(x) = d_i \qquad (1 \leq i \leq m)$$

where $c \in R^n$ and $d = (d_1,\ldots,d_m) \in R^m$, we see that for every sufficiently small $(c,d) \in R^{n+m}$ the system of equations

$$F(x,y) = (c,d), \quad (x,y) \in B_{\delta^*}(x^*) \times R^m$$

has a unique solution. This implies that for some positive $\delta_1 \leq \min\{\delta_0,\delta^*\}$ F is one-to-one in $B_{\delta_1}(z^*)$. Hence, by Lemma 2.3, for some $s \in \{-1,+1\}$ and any $\delta \in (0,\delta_1]$, we have

$$\deg(F,B_\delta(z^*),0) = s$$

and

$$\text{sign}\det DF(z;\sigma) = s \text{ or } 0 \text{ for every } z \in \sigma \cap B_{\delta_1}(z^*) \text{ and } \sigma \in K^* .$$

If $\text{sign}\det DF(z^*;\sigma) = s$ for every $\sigma \ni z^*$ then, by Theorem 3.3, F is locally nonsingular at z^*.

We assume on the contrary that $\det DF(z^*;\sigma^*) = 0$ for some $\sigma^* \in K^*$ containing z^*, and show that for any positive numbers $\delta \leq \min\{\delta_0,\delta^*\}$ and $\alpha \leq \alpha_0$ there exists $g \in F^*$ with $\mathrm{norm}\,(g,B_{\delta^*}(x^*)) \leq \alpha$ such that $F+G$ is not one-to-one in $B_\delta(x^*)$. Then we can see that x^* is not s-stable (w.r.t. F^*) because if two distinct $(x^j,y^j) \in B_\delta(z^*)$ $(j = 1,2)$ satisfy

$$F(x^j,y^j) + G(x^j,y^j) = (c,d)$$

for some sufficiently small $(c,d) \in R^{n+m}$ and if we define

$$g_0'(x) = g_0(x) - c^T x,$$

$$g_i'(x) = g_i(x) + d_i \qquad (1 \leq i \leq m)$$

for all $x \in R^n$ then x^1 and x^2 are two distinct stationary solution to Pl(f + g').

Let $0 < \delta \leq \min\{\delta_0,\delta^*\}$ and $\alpha > 0$. Recall that $z^* \in \sigma^* \in K^*$ and $\det DF(z^*;\sigma^*) = 0$. Let $J \subset \{\ell+1,\ldots,k_1\}$ be such that $\sigma^* = \tau(J)$, and k_0 be the dimension of the space $W = \{v \in R^n : \nabla f_i(x^*)^T v = 0, i \in \{1,\ldots,\ell\} \cup J\}$. If $k_0 = 0$ then, by Theorem 3.5, we have $\det DF(z^*;\tau(J)) > 0$, a contradiction. Hence $k_0 \geq 1$. Let B be an $n \times k_0$ matrix whose columns form a basis of the space W. By Theorem 3.5, we see

$$\det B^T L(z^*) B = 0 ,$$

where $L(z^*)$ is defined by (3-6). Let $\bar{s} = \{-1,1\} \backslash \{s\}$. By Lemma 3.4, there exists an $n \times n$ symmetric matrix Q such that

$$\text{sign}\det B^T(L(z^*) + \gamma Q)B = \bar{s} \text{ for all } \gamma > 0 .$$

Define, for all $x \in R^n$,

$$g_0(x) = \frac{1}{2}(x - x^*)^T \gamma Q(x - x^*)$$

and

$$g_i(x) = 0 \qquad (1 \leq i \leq m) .$$

Then, by Theorem 3.5, we see

$$\begin{aligned} &\operatorname{sign} \det[DF(z^*;\sigma^*) + DG(z^*;\sigma^*)] \\ &= \operatorname{sign} \det B^T(L(z^*) + \gamma Q)B \qquad \text{(4-9)} \\ &= \bar{s} \quad \text{for every} \quad \gamma > 0 . \end{aligned}$$

Since F is one-to-one in $B_\delta(z^*)$ and $\deg(F, \operatorname{int} B_\delta(z^*), 0) = s$, using the homotopy invariance theorem (6.2.2 of Ortega and Rheinbold [24]), we have

$$\deg(F + G, \operatorname{int} B_\delta(z^*), 0) = s \quad \text{for every sufficiently small} \quad \gamma ,$$

which together with (4-9) implies that $F + G$ is not one-to-one in $B_\delta(z^*)$ for every sufficiently small $\gamma > 0$ (see Lemma 2.3). Q.E.D.

Corollary 4.3. Let $z^* = (x^*,y^*)$ be a stationary point to $Pl(f)$. Assume that Condition 1.1 holds. Then the following three are equivalent:

(4-10) F is locally nonsingular at z^*.

(4-11) x^* is a s-stable (w.r.t. $\mathcal{F}$) stationary solution to $Pl(f)$.

(4-12) x^* is a s-stable (w.r.t. $\mathcal{F}^*$) stationary solution to $Pl(f)$.

Proof. We have shown that (4-10) implies (4-11) (Theorem 4.1) and that (4-12) implies (4-10) (Theorem 4.2). By the definition of the strong stability, (4-11) obviously implies (4-12). Q.E.D.

5. S-STABLE (w.r.t. $\mathcal{F}$) LOCAL MINIMUM SOLUTIONS.

In this section, we focus our attention to the set of local minimum solutions to $Pl(f)$ which are s-stable (w.r.t. $\mathcal{F}$). We will be concerned with the following two conditions:

Condition 5.1. $L(z^*)$ is positive semi-definite on the space $\{w \in R^n : \nabla f_i(x^*)^T w = 0$ for all $i \in \{1,\ldots,m\}$ such that $f_i(x^*) = 0\}$.

Condition 5.2. $L(z^*)$ is positive definite on the space $\{w \in R^n : \nabla f_i(x^*)^T w = 0$ for all $i \in \{1,2,\ldots,\ell\} \cup \{j : y_j^* > 0, \ell + 1 \leq j \leq m\}\}$.

Lemma 5.3. Suppose that $z^* = (x^*,y^*) \in R^{n+m}$ is a stationary point to $Pl(f)$. If x^* is a local minimum solution to $Pl(f)$ satisfying Condition 1.1 then $z^* = (x^*,y^*)$ satisfies Condition 5.1. If $z^* = (x^*,y^*)$ satisfies Condition 5.2 then x^* is an isolated local minimum solution to $Pl(f)$.

Proof. See Sections 2.2 and 2.3 of Fiacco and McCormick [15].

The following two theorems characterize s-stable (w.r.t. $\mathcal{F}$) local minimum solutions to $Pl(f)$.

Theorem 5.4. Let $z^* = (x^*,y^*)$ be a stationary point to $Pl(f)$. Assume that Conditions 1.1. and 5.2 hold. Then x^* is s-stable (w.r.t. $\mathcal{F}$).

Proof. In view of Theorem 3.3 and Corollary 4.3, it suffices to show

$$\det DF(z^*;\tau(J)) > 0 \quad \text{for all} \quad J \quad \text{such that} \quad J_0 \subset J \subset J_1 ,$$

where J_0 and J_1 are given by (3-1). Let $J_0 \subset J \subset J_1$. If the index set $\{1,\ldots,\ell\} \cup J$ has n elements, Theorem 3.5 ensures that $\det DF(z^*;\tau(J)) > 0$. Suppose that the set $\{1,\ldots,\ell\} \cup J$ has $k < n$ elements. Then the subspace W of R^n defined by

$$W = \{w \in R^n : \nabla f_i(x^*)^T w = 0 \quad \text{for all} \quad i \in \{1,\ldots,\ell\} \cup J\}$$

has the dimension $n - k \geq 1$. Let B be an $n \times (n - k)$ matrix whose columns form a basis of the space W. Since W is a subspace of

$$\{w \in R^n : \nabla f_i(x^*)^T w = 0 \quad \text{for all} \quad i \in \{1,\ldots,\ell\} \cup J_0\} ,$$

it follows from Condition 5.2 that the $(n - k) \times (n - k)$ matrix $B^T L(z^*) B$ is positive

definite, which implies $\det B^T L(z^*) B > 0$. Therefore, by Theorem 3.5, we obtain $\det DF(z^*;\tau(J)) > 0$. Q.E.D.

Theorem 5.5. Let $z^* = (x^*,y^*)$ be a stationary point to P1(f). Assume that x^* is s-stable (w.r.t. F) and that Conditions 1.1. and 5.1 hold. Then $z^* = (x^*,y^*)$ satisfies Condition 5.2.

Proof. By Corollary 4.3, $F : |K^*| \to R^{n+m}$ is locally nonsingular at z^*. Let J_0 and J_1 be the index set defined by (3-1). For simplicity of notations, we assume that $J_0 = \{\ell+1,\ldots,k_0\}$ and $J_1 = \{\ell+1,\ldots,k_1\}$ for some k_0 and k_1 such that $\ell \leq k_0 \leq k_1 \leq n$. If $k_0 = k_1 = n$ then the dimension of the space

$$\{w \in R^n : \nabla f_i(x^*)^T w = 0 \text{ for all } i \in \{1,\ldots,\ell\} \cup J_0\}$$

is zero and Condition 5.2 holds. Suppose $k_0 \leq n-1$. For each $k \in \{k_0,\ldots,k_1\}$, define

$$W_k = \{w \in R^n : \nabla f_i(x^*)^T w = 0 \ (1 \leq i \leq k)\} .$$

Then each space W_k has dimension $n-k$ and $W_{k+1} \subset W_k$ $(k_0 \leq k \leq k_1 - 1)$. Choose an $n \times (n-k_0)$ matrix C such that the first $(n-k)$ columns of C form a basis of $W_k (k_0 \leq k \leq \min\{k_1, n-1\})$. We shall show that the $(n-k_0) \times (n-k_0)$ symmetric matrix $C^T L(z^*) C$ is positive definite. Let B^k be the $n \times (n-k)$ matrix consisting of the first k columns of C $(k_0 \leq k \leq \min\{k_1, n-1\})$. Assume that $k_1 \leq n-1$. It follows from Condition 5.1 that the $(n-k_1) \times (n-k_1)$ symmetric matrix

(5-1) $$(B^{k_1})^T L(z^*) B^{k_1}$$

is positive semi-definite. On the other hand, by the local nonsingularity, the matrix (5-1) is nonsingular (see Corollary 3.6). Hence the matrix (5-1) is positive definite. By Corollary 3.6, for every $k \in \{k_0,\ldots,k_1\}$

(5-2) $$\det (B^k)^T L(z^*) B^k > 0$$

Thus we have shown that all the leading principal minors of the symmetric matrix $C^T L(z^*) C$ are positive. Hence $C^T L(z^*) C$ is positive definite.

Finally we deal with the case where $k_1 = n$, i.e., $J_1 = \{1,\ldots,n\}$. In this case we see $\dim W_{k_1} = 0$. By Theorem 3.5, $\det DF(z^*;\tau(J_1)) > 0$, and by Theorem 3.3, $\det DF(z^*;\tau(\{1,\ldots,k\})) > 0$ for all $k \in \{k_0,\ldots,n\}$. Thus (5-2) holds for every $k \in \{k_0,\ldots,n-1\}$ (Theorem 3.5). Therefore all the leading principal minors of $C^T L(z^*) C$ are positive. Q.E.D.

As a direct consequence of Lemma 5.3 and Theorem 5.5, we have:

Corollary 5.6. Let $z^* = (x^*,y^*)$ be a stationary point to P1(f). Suppose that x^* is s-stable (w.r.t. F) and that Condition 1.1 holds. Then the following three are equivalent:

(5-3) x^* is an isolated local minimum solution to P1(f).

(5-4) $z^* = (x^*,y^*)$ satisfies Condition 5.1.

(5-5) $z^* = (x^*,y^*)$ satisfies Condition 5.2.

6. AN APPLICATION TO A PARAMETRIC PROGRAM.

Let T be a closed subset of R^q, h_i a map from $R^n \times T$ into R $(0 \le i \le n)$ and $h = (h_0, h_1, \ldots, h_m)$. Throughout this and next sections, we assume that $h(\cdot,t) \in \mathcal{F}$ for each $t \in T$ and that $\partial h_i(x,t)/\partial x_j$, $\partial^2 h_i(x,t)/\partial x_j \partial x_k$ $(0 \le i \le m, 1 \le j \le n, 1 \le k \le n)$ are continuous with respect to $(x,t) \in R^n \times T$. We will deal with the following parametric nonlinear program:

P2(t) $\qquad$ minimize $h_0(x,t)$

$\qquad\qquad$ subject to $x \in Y(t)$,

where

$$Y(t) = \{x \in R^n : h_i(x,t) = 0 \quad (1 \le i \le \ell)$$
$$h_j(x,t) \le 0 \quad (\ell + 1 \le j \le m)\} .$$

and $t \quad T$ is a parameter vector. The Kuhn-Tucker stationary condition to P2(t) can be written as

$$H(x,y,t) = 0 , \tag{6-1}$$

where

$$H(x,y,t) = \begin{bmatrix} \nabla h_0(x,t) + \sum_{i=1}^{\ell} y_i \nabla h_i(x,t) + \sum_{j=\ +1}^{m} y_j^+ \nabla h_j(x,t) \\ -h_i(x,t) \\ \vdots \\ -h_\ell(x,t) \\ y_{\ell+1}^- - h_{\ell+1}(x,t) \\ \vdots \\ y_m^- - h_m(x,t) \end{bmatrix} \tag{6-2}$$

for every $(x,y,t) \in R^{n+m} \times T$, where $\nabla h_i(x,t) = (\partial h_i(x,t)/\partial x_1, \ldots, \partial h_i(x,t)/\partial x_n)^T$. Note that for each $t \in T$ the map $H(\cdot,\cdot,t) : R^{n+m} \to R^{n+m}$ is PC^1 on the subdivision K^* defined by (2-3) and (2-4). Let

$\Sigma = \{(x,t) \in R^n \times T : x$ is a stationary solution to P2(t)$\}$,

(6-3) $\quad \Sigma^s = \{(x,t) \in R^n \times T : x$ is a s-stable (w.r.t. $\mathcal{F}$) stationary solution to P2(t)$\}$,

$\Sigma^* = \{(x,t) \in R^n \times T : x$ is a local minimum solution to P2(t)$\}$.

obviously $\Sigma^s \subset \Sigma$. We will assume the following constraint qualification which ensures that $\Sigma^* \subset \Sigma$.

Condition 6.1. If $(x,t) \in \Sigma \cup \Sigma^*$ then the set of vectors $\nabla h_i(x,t)$ such that $h_i(x,t) = 0$ $(1 \le i \le m)$ is linearly independent.

Theorem 6.2. Let $z^* = (x^*,y^*)$ be a stationary point to $P2(t^*)$. Assume that the map $H(\cdot,t^*) : |K^*| \to R^{n+m}$ is locally nonsingular at $z^* = (x^*,y^*)$. Then there exist open neighborhoods U of z^*, V of t^* and a continuous map $z : V \cap T \to U$ such that $z(t)$ is a stationary point to P2(t) for all $t \in V \cap T$ and that if $z' \in U$ is a stationary point to P2(t) for some $t \in V \cap T$ then $z' = z(t)$.

Proof. By Theorem 4.1., x^* is a s-stable (w.r.t. $\mathcal{F}$) stationary solution to $P2(t^*)$. Hence there exist positive numbers δ and β such that if $t \in B_\beta(t^*)$ then $B_\delta(x^*)$ contains a unique stationary solution to P2(t), which we will denote by $x(t)$. Let $y(t) \in R^m$ be the Lagrange multiplier vector associated with the stationary solution $x(t)$ to P2(t); $(x(t),y(t),t)$ satisfies (6-1). Let $z(t) = (x(t),y(t))$. The continuity of the map z for every t sufficiently close to t^* follows from the fact $z(t)$ is a locally nonsingular point of $H(\cdot,t) : |K^*| \to R^{n+m}$, which implies that $x(t)$ is a s-stable (w.r.t. $\mathcal{F}$) stationary solution to P2(t), for every t sufficiently close to t^*. Q.E.D.

Let $V^1 \subset V^2 \subset R^p$. We say that V^1 is open (or closed) relative to V^1 if $V^1 = V^2 \cap U$ for some open (or closed) subset U of R^p

Theorem 6.3. Assume that Condition 6.1 holds. Then $\Sigma^s \cap \Sigma^*$ is open and closed relative to Σ^s.

Proof. First we show $\Sigma^s \cap \Sigma^*$ is open relative to Σ^s. Let $(x^*,t^*) \in \Sigma^s \cap \Sigma^*$. By Condition 6.1, there is a unique $y^* \in R^m$ such that (x^*,y^*,t^*) satisfies (6-1) and that the map $H(\cdot,t^*) : |K^*| \to R^{n+m}$ is locally nonsingular at $z^* = (x^*,y^*)$. It suffices to show that if a solution (x,y,t) of (6-1) is sufficiently close to (x^*,y^*,t^*) then $(x,t) \in \Sigma^*$. Let $I_0 = \{1,\ldots,\ell\} \cup \{j : y_j^* > 0, \ell + 1 \le j \le m\}$. By Corollary 5.6, the $n \times n$ symmetric matrix $H(z^*,t^*)$ is positive definite on the space W, where

$$W = \{w \in R^n : \nabla h_i(x^*,t^*)^T w = 0, i \in I_0\},$$

$$N(z,t) = \nabla^2 h_0(x,t) + \sum_{i=1}^{\ell} y_i \nabla^2 h_i(x,t) + \sum_{j=\ell+1}^{m} y_j^+ \nabla^2 h_j(x,t)$$

for all $(z,t) = (x,y,t) \in R^{n+m} \times T$ and $\nabla^2 h_i(x,t)$ denote the Hessian matrix of the map $h_i(\cdot,t) : R^n \to R^n$ at x. Hence if a solution $(z,t) = (x,y,t)$ to (6-1) is sufficiently close to (x^*,y^*,t^*) then $I_0 \subset I$ and the matrix $N(z,t)$ is positive definite on the space $\{w \in R^n : \nabla h_i(x,t)^T w = 0\ (i \in I)\}$, where $I = \{1,\dots,\ell\} \cup \{j : y_j > 0, \ell+1 \leq j \leq m\}$; hence $(x,t) \in \Sigma^*$ by Lemma 5.3. Thus we have shown that $\Sigma^s \cap \Sigma^*$ is open relative to Σ^s.

To prove that $\Sigma^s \cap \Sigma^*$ is closed relative to Σ^s, we consider a sequence $\{(x^p,t^p) \in \Sigma^s \cap \Sigma^*\}$ which converges to some $(x^*,t^*) \in \Sigma^s$, and we shall show $(x^*,t^*) \in \Sigma^*$. Corresponding to each $(x^p,t^p) \in \Sigma^s \cap \Sigma^*$, there is a unique $y^p \in R^m$ such that (x^p,y^p,t^p) satisfies (6-1). Also for some unique $y^* \in R^m$, (x^*,y^*,t^*) satisfies (6-1). Let $J_1 = \{i : h_i(x^*,t^*) = 0, \ell+1 \leq i \leq m\}$. Taking an appropriate subsequence if necessary, we may assume that

$$J = \{j : h_j(x^p,t^p) = 0, \ell+1 \leq i \leq m\}$$

for some $J \subset \{\ell+1,\dots,m\}$ and all p. By the continuity of the maps $h_i (1 \leq i \leq m)$, we see $J_1 \supset J$. Hence Condition 6.1 ensures that $\{y^p\}$ converges to y^*.

For every p, define

$$W_p = \{w \in R^n : \nabla h_i(x^p,t^p)^T w = 0, i \in \{1,\dots,\ell\} \cup J\}$$

and

$$W_* = \{w \in R^n : \nabla h_i(x^*,t^*)^T w = 0, i \in \{1,\dots,\ell\} \cup J\}$$

By Lemma 5.3, the $n \times n$ symmetric matrix $N(z^p,t^p)$, where $z^p = (x^p,y^p)$, is positive semi-definite on the space W_p. Taking the limit as $p \to +\infty$, we obtain that $N(z^*,t^*)$ is positive semi-definite on W_*, which together with $J \subset J_1$ implies that $N(z^*,t^*)$ is positive semi-definite on the space

$$\{w \in R^n : \nabla h_i(x^*,t^*)^T w = 0, i \in \{1,\dots,\ell\} \cup J_1\}.$$

The desired result follows from Corollary 5.6. Q.E.D.

It is easily verified that if Condition 6.1 holds then Σ^s is open relative to Σ. Hence $\Sigma^s \cap \Sigma^*$ is open relative to Σ under Condition 6.1.

Corollary 6.4. Suppose that Condition 6.1 holds. Let S be a connected subset of Σ^s which contains (x^*,t^*) such that x^* is a local minimum solution to P2(t^*). Then x is a local minimum solution to P2(t) for all $(x,t) \in S$.

Proof. By Theorem 6.3, there exist an open subset U of R^{n+q} and a closed subset V of R^{n+q} such that $\Sigma^s \cap \Sigma^* = \Sigma^s \cap U = \Sigma^s \cap V$. Hence

$$(x^*,t^*) \in S \cap \Sigma^* = S \cap \Sigma^s \cap \Sigma^* = S \cap U = S \cap V,$$

which implies that the nonempty subset $S \cap \Sigma^*$ of S is open and closed relative to S. Since S is connected, we obtain $S = S \cap \Sigma^*$. Q.E.D.

7. AN APPLICATION TO A CLASS OF CONTINUOUS DEFORMATION METHODS.

To solve P1(f), we artificially construct another nonlinear program P1(g) which has a trivial stationary solution x^0 and a *parametric nonlinear program* P2(t) ($t \in [0,1]$) which continuously deform P1(g) to P1(f), i.e., $h(\cdot,0) = g$ and $h(\cdot,1) = f$. Then we start from $(x^0,0)$ and trace the connected component $S^0 \subset \Sigma$ which contains $(x^0,0)$ to attain a stationary solution x^1 to P2(1). Some methods (Eaves [9, 11], Kojima [17, 19], Lemke [21], Saigal [30], Todd [33]) which were developed in the fixed point and complementarity theory are based on the above idea.

We can prove:

__Theorem 7.1.__ Let $T = [0,1]$. Suppose Condition 6.1 holds. Let x^0 be a s-stable (w.r.t. *F*) local minimum solution to P2(0) and $S^0 \subset \Sigma$ be a compact connected set which contains $(x^0,0)$. Assume that $S^0 \subset \Sigma^s$. Then S^0 can be written as

$$S^0 = \{(x(t),t) : 0 \leq t \leq t^*\}$$

for some $t^* \in [0,1]$ and some continuous map $x : [0,t^*] \to R^n$. Furthermore, each $x(t)$ is an isolated local minimum solution to P2(t).

__Proof.__ The first part directly follows from Theorem 6.2 and the second from Corollary 6.4. Q.E.D.

__Remark.__ By using Sard's theorem (5.2.5 of Ortega and Rheinboldt [24]), we can easily verify that for almost all $(c,d) \in R^{n+m}$ the set

$$\{(x,y,t) \in R^{n+m} \times [0,1] : H(x,y,t) = (c,d)\}$$

consists of disjoint piecewise smooth one manifolds. See also Alexander [1], Chow, Mallet-Paret and York [6], and Garcia and Gould [16].

__Corollary 7.2.__ Let T, x^0, S^0, t^* and x be as in Theorem 7.1. *If* $h : R^{n+1} \to R^{1+m}$ has the form

$$h_0(x,t) = (1-t)g_0(x) + tf_0(x)$$
$$h_i(x,t) = f_i(x) \qquad (1 \leq i \leq m)$$

for every $(x,t) \in R^n \times T$, then $x(t) \in X(f)$ for all $t \in [0,t^*]$ and $f_0(x(t))$ is monotone nonincreasing with respect to $t \in [0,t^*]$.

__Proof.__ It *suffices to show* that if $\epsilon > 0$ is sufficiently small and $0 \leq t < t + \epsilon \leq t^*$ then $f_0(x(t+\epsilon)) \leq f_0(x(t))$. Let $t \in [0,t^*)$. Since $x(t)$ is an isolated local minimum solution to P2(t), there exists a positive number δ such that

$$h_0(x(t),t) < h_0(x',t) \quad \text{for all } x' \in U,$$

where $U = \{x' \in X(f) : x' \in B_\delta(x(t)), x' \neq x(t)\}$. Specifically, we have

$$h_0(x(t),t) < h_0(x',t) \quad \text{for all } x' \in V,$$

where $V = \{x' \in U : \|x' - x(t)\| = \delta\}$. By the continuity of h_0, if $\epsilon > 0$ is sufficiently small then

$$h_0(x(t),t+\epsilon) < h_0(x',t+\epsilon) \quad \text{for all } x' \in V$$

Hence for every sufficiently small $\epsilon > 0$, the program:

$$\text{minimize} \quad h_0(x',t+\epsilon)$$
$$\text{subject to} \quad x' \in X(f) \cap B_\delta(x(t))$$

has a minimum solution in the interior of $B_\delta(x(t))$, which must coincide with $x(t+\epsilon)$. Therefore, we obtain that, for every sufficiently small $\epsilon > 0$,

$$h_0(x(t),t) \leq h_0(x(t+\epsilon),t),$$
$$h_0(x(t+\epsilon),t+\epsilon) \leq h_0(x(t),t+\epsilon),$$

from which the desired result follows. Q.E.D.

8. CONCLUDING REMARKS.

As stated in the Introduction, the Kuhn-Tucker stationary condition can be also formulated as a system of generalized equations. The local nonsingularity of the map $F : |K^{*}| \to R^{n+m}$ defined by (1-2) is equivalent to the strong regularity (see Robinson [26, 27]) for the system of generalized equations associated with the Kuhn-Tucker stationary condition. Theorem 6.2 follows from Theorem 2.1 of Robinson [27]. Also Theorems 3.3 and 5.4 have close relations with Theorems 4.1 and 3.1 of [27], respectively.

The result in Theorem 7.1 was recently shown by Saigal [30] for a special case where P2(t) has no constraints (i.e., $m = 0$). See also Saari and Saigal [28].

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For each continuously twice differentiable map $f = (f_0, f_1, \ldots, f_m) : R^n \to R^{1+m}$, we define a nonlinear program:

$$\text{Pl}(f) \qquad \text{minimize } f_0(x)$$
$$\text{subject to } f_i(x) = 0 \quad (1 \leq i \leq \ell),$$
$$f_j(x) \leq 0 \quad (\ell + 1 \leq j \leq m). \quad \text{(continued)}$$

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A (Kuhn-Tucker's) stationary solution x to $P1(f)$ is said to be strongly stable if there exists an open neighborhood U of x such that each open neighborhood $V \subset U$ of x contains a stationary solution to a perturbed problem $P1(f + g)$ which is unique in U whenever $g_i(x)$, $\partial g_i(x)/\partial x_j$ and $\partial^2 g_i(x)/\partial x_j \partial x_k$ $(0 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq k \leq n)$ are sufficiently small for all x in U. We will give conditions on the gradient vectors and the Hessian matrices of $f_i (0 \leq i \leq m)$ which characterize the strong stability. These conditions are then applied to a parametric nonlinear program:

$$
\begin{array}{lll}
P2(t) \quad & \text{minimize} & h_0(x,t) \\
& \text{subject to} & h_i(x,t) = 0 \qquad (1 \leq i \leq \ell), \\
& & h_j(x,t) \leq 0 \qquad (\ell + 1 \leq j \leq m),
\end{array}
$$

where t is a parameter vector varying in a closed subset T of R^q. Let Σ^s be the set of points (x,t) in $R^n \times T$ such that x is a strongly stable stationary solution to $P2(t)$. Under a certain constraint qualification and the continuity and the differentiability of the map $h : R^n \times T \to R^{1+m}$, we will establish that if S is a connected subset of Σ^s and if x^* is a local minimum solution to $P2(t^*)$ for some $(x^*,t^*) \in S$ then x is a local minimum solution to $P2(t)$ for all $(x,t) \in S$. Finally this result is applied to showing some interesting properties of a class of methods developed in the fixed point and complementarity theory.